

APPENDIX III¹

Dialogues: for first order logic and for modal positive free logic

Formal dialogues

Let us see what is at stake in dialogical logic by reconstructing in dialogical terms the notion of validity in First-order logic.² We first define a language $L[\tau]$; this language will basically be obtained from First-order logic (of vocabulary τ) by adding certain metalogical symbols.

We introduce special *force symbols* $?$ and $!$.

An *expression* of $L[\tau]$ is either a formula of $\mathbf{FO}[\tau]$, or one of the following strings:

$$L, R, \vee, \forall x_i/k_j \text{ or } \exists x_i/k_j$$

where x_i is any variable and k_j any constant. We refer to the latter type of expressions as *attack markers*.

In addition to expressions and force symbols, for $L[\tau]$ we have available *labels* \mathbf{O} and \mathbf{P} , standing for the players (*Proponent*, *Opponent*) of dialogues.

Every expression e of $L[\tau]$ can be augmented with labels \mathbf{P} or \mathbf{O} on the one hand, and by the force symbols $?$ and $!$ on the other, so as to yield the strings

$$\mathbf{P}!\text{-}e, \mathbf{O}!\text{-}e, \mathbf{P}?\text{-}e, \mathbf{O}?\text{-}e$$

These strings are said to be (*dialogically*) *signed expressions*. Their role is to signify that in the course of a dialogue, the move corresponding to the expression e is to be made by \mathbf{P} or \mathbf{O} , respectively, and that the move is made as a defence (!) or an attack (?). We will use \mathbf{X} and \mathbf{Y} as variables for \mathbf{P} and \mathbf{O} , always assuming $\mathbf{X} \neq \mathbf{Y}$.

Particle rules

An *argumentation form* or *particle rule* is an abstract description of the way a formula, according to its outmost form, can be criticized, and how to answer the critique. It is abstract in the sense that this description can be carried out without reference to a specified context. In dialogical logic, these rules are said to state the *local semantics*, for they show how the game runs locally: what is at stake is only the critique and the answer corresponding to a given logical constant, rather than the whole context where the logical constant is embedded.³ The particle rules fix the dialogical semantics of the logical constants of $L[\tau]$ in the following way:

¹ The present appendix is based on the first 6 pages of Rahman 2009b.

² This version is essentially from Rahman/Tulenheimo [2006]. For somewhat different accounts, see Rahman \Keiff 2005, Fontaine/Redmond 2008, and Keiff 2009.

³ There can be no particle rule corresponding to atomic formulae. But it is possible to add a set of *Opponent's* initial concessions to the particle rules. This is done in 'material dialogues' (see Rahman/Tulenheimo [2006]).

Assertion	Attack	Defence
$\mathbf{X}\text{-!-}A \vee B$	$\mathbf{Y}\text{-?}-\vee$	$\mathbf{X}\text{-!-}A$ or $\mathbf{X}\text{-!-}B$ (the defender chooses)
$\mathbf{X}\text{-!-}A \wedge B$	$\mathbf{Y}\text{-?}\wedge 1$ or $\mathbf{Y}\text{-?}\wedge 2$ (the challenger chooses)	$\mathbf{X}\text{-!-}A$ respectively $\mathbf{X}\text{-!-}B$
$\mathbf{X}\text{-!-}A \rightarrow B$	$\mathbf{Y}\text{-!-}A$	$\mathbf{X}\text{-!-}B$
$\mathbf{X}\text{-!-}\neg A$	$\mathbf{Y}\text{-!-}A$	\otimes No defence possible. Only a counterattack is available
$\mathbf{X}\text{-!-}\forall xA$	$\mathbf{Y}\text{-?}-\forall x/k$ For any k available to \mathbf{Y}	$\mathbf{X}\text{-!-}A[x/k]$ For any k chosen earlier by \mathbf{Y}
$\mathbf{X}\text{-!-}\exists xA$	$\mathbf{Y}\text{-?}-\exists$	$\mathbf{X}\text{-!-}A[x/k]$ For any k available to \mathbf{Y}

In the diagram, $A[x/k]$ stands for the result of substituting the constant k for every occurrence of the variable x in the formula A .

A more thorough way to stress the sense in which the particle rules determine local semantics is to see these rules as defining the notion of *state* of a (structurally not yet determined) game.

DEFINITION [*State of a dialogue*]:

Let A be a formula of $\mathbf{FO}[\tau]$, and let a countable set $\{k_0, k_1, \dots\}$ of individual constants be fixed. A *state of the dialogue* $D(A)$ corresponding to the formula A is a quintuple

$\langle B, X, \mathcal{I}, e, \sigma \rangle$ such that:

- B is a (proper or improper) subformula of A .
- X, \mathcal{I}, e is a dialogically signed expression. Thus, X is either **O** or **P** and , $\mathcal{I} \in \{?, !\}$, and $e \in \mathcal{L}[\tau]$.
- $\sigma : \text{Free}[B] \rightarrow \{k_0, k_1, \dots\}$ is a function mapping the free variables of B to individual constants.

➤ The component e is either a formula of $\mathbf{FO}[\tau]$, or an attack marker. We stipulate that in the former case, always $e := B$

Given a force \mathcal{I} , let us write \mathcal{I}' , for the opposite force, i.e. let

$$\mathcal{I}' \in \{?, !\} \setminus \mathcal{I}.$$

Each state $\langle B, X, Y, e, \sigma \rangle$ has an associated *role* assignment, indicating which player occupies the role of *challenger* and which the role of defender. In fact, the role assignment is a function $\rho : \{\mathbf{P}, \mathbf{O}\} \times \{?, !\}$ such that $\rho(\mathbf{X}) = Y$ and $\rho(\mathbf{Y}) = X$.

We say that the state $\langle B_2, X_2, Y_2, e_2, \sigma_2 \rangle$ is *reachable* from state $\langle B_1, X_1, Y_1, e_1, \sigma_1 \rangle$ if it is a result of X_1 making a move in accordance with the appropriate particle rule in the role Y_1 . If the role is that of *challenger* ($Y_1 = ?$), the player states an attack, whereas if the role is that of *defender* ($Y_1 = !$), the player poses a defence.

Let us take a closer look at the transitions from one state to another. Particle rules determine which state S_2 of a dialogue is reachable from a given other state S_1 . Notice that the player who defends need not be the same at both states. In order for state S_2 to be reachable from state $S_1 = \langle B, X, Y, e, \sigma \rangle$, it must satisfy the following.

- *Particle rule for negation:* If $B = e$, $Y = !$ and B is of the form $\neg C$, then

$$S_2 = \langle C, Y, !, C, \sigma \rangle$$

So if \mathbf{P} is *defender* of $\neg C$ at S_1 , then \mathbf{O} is *defender* of C at S_2 , and \mathbf{P} will challenge (counterattack) C ; and dually, if \mathbf{P} is *challenger* of $\neg C$ at S_1 .

Notice that here state S_2 involves the claim that C can be defended; however, this claim has been asserted in the course of an attack, and the whole move from S_1 to S_2 counts as an attack on the initial negated formula, i.e. an attack on C . Actually this follows from the fact that at S_2 , the roles of the players are inverted as compared with S_1 . Counterattack may yield from S_2 a further state, $S_3 = \langle C, X, ?, *, \sigma \rangle$, where C is the formula considered, and the attack pertains to the relevant logical constant of C , for which $*$ is a suitable attack marker determined by the logical form of C .

- *Particle rule for conjunction:* If $B = e$, $Y = !$ and B is of the form $C \wedge D$, then

$$S_2 = \langle C, X, !, C, \sigma \rangle \text{ or } S_2 = \langle D, X, !, D, \sigma \rangle$$

according to the choice of the *challenger* between the attacks $?-L$ and $?-R$. (Here the challenger is Y : Y 's role is $?$ here.)

- *Particle rule for disjunction:* If $B = e$, $Y = !$ and B is of the form $C \vee D$, then

$$S_2 = \langle C, X, !, C, \sigma \rangle \text{ or } S_2 = \langle D, X, !, D, \sigma \rangle$$

according to the choice of the *defender*, reacting to the attack $?- \vee$ of the challenger. (Here the defender is X : X 's role is $!$ here.)

- *Particle rule for conditional:* If $B = e$, $Y = !$ and B is of the form $C \rightarrow D$, then

$$S_2 = \langle C, Y, !, C, \sigma \rangle$$

and, further, state

$$S_3 = \langle D, X, !, D, \sigma \rangle$$

is reachable from S_2 . So if \mathbf{P} is the *defender* of $C \rightarrow D$ at S_1 , and hence \mathbf{O} is the *defender* of C at S_2 , it is \mathbf{P} who will be the *defender* of D at S_3 .

To attack a conditional amounts to being prepared to defend its antecedent, and so it should be noticed that the defence of C at state S_2 counts as an attack. If \mathbf{P} is the *defender* of $C \rightarrow D$ at S_1 , then at state S_3 reachable from S_2 , either \mathbf{P} may defend D , or else \mathbf{P} may counterattack C , thus yielding a further state, $S_4 = \langle C, X, ?, *, \sigma \rangle$, where C is the formula considered, and the attack pertains to the relevant logical constant of C , for which $*$ is a suitable attack marker determined by the logical form of C .

- *Particle rule for universal quantifier*: If $B=e$, $\mathcal{I} \models !$ and B is of the form $\forall x D x$, then

$$S_2 = \langle D x, X, !, D x, \sigma[x/k_i] \rangle$$

where k_i is the constant chosen by the *challenger* (who here is Y) as a response to the attack $?\text{-}\forall x/k_i$.

As usual, the notation ' $\sigma[x/k_i]$ ' stands for the function that is otherwise like σ , but maps the variable x to k_i . Hence if σ is already defined on x , $\sigma[x/k_i]$ is the result of reinterpreting x by k_i otherwise it is the result of extending σ by the pair (x, k_i)

- *Particle rule for existential quantifier*: If $B=e$, $\mathcal{I} \models !$ and B is of the form $\exists x D x$, then

$$S_2 = \langle D x, X, !, D x, \sigma[x/k_i] \rangle$$

where k_i is the constant chosen by the *defender* (who here is X) reacting to the attack $?\text{-}\exists x$ of the *challenger* (Y).

Structural rules

As we analyze dialogues, we will make use of the following notions: *dialogue*, *dialogical game*, and *play of a dialogue*. It is very important to keep them conceptually distinct. Dialogical games are sequences of dialogically signed expressions, i.e. expressions of the language $L[\tau]$ equipped with a pair of labels, $\mathbf{P}\text{-!}$, $\mathbf{O}\text{-!}$, $\mathbf{P}\text{-?}$, or $\mathbf{O}\text{-?}$. The labels carry information about how the dialogue proceeds. Dialogical games are a special case of plays: all dialogical games are plays, but not all plays are dialogical games. However, all plays *are* sequences of dialogical games. Finally, dialogues are simply sets of plays.

A complete dialogue is determined by game rules. They specify how dialogical games in particular, and plays of dialogues in general, are generated from the thesis of the dialogue. Particle rules are among the game rules, but in addition to them there are so-called *structural rules*, which serve to specify the general organization of the dialogue.

Different types of dialogues have different kinds of structural rules. *When the issue is to test validity* - as it is for the dialogues considered in the present paper - a dialogue can be thought of as a tree, whose (maximal) branches are (finished) plays relevant for establishing the validity of the thesis. The structural rules will be chosen so that *Proponent* succeeds in defending the thesis against all allowed critique of *Opponent* if, and only if, the thesis is valid in the standard sense of the term ('true in every model'). In dialogical logic the existence of such a winning strategy for *Proponent* is typically taken as the definition of validity; however, this dialogical definition indeed captures the standard notion (see the discussion in connection with the definition of validity below).

Each split into two branches - into two plays - in a dialogue tree should be considered as the outcome of a propositional choice made by *Opponent*. Any choice by **O** in defending a disjunction, attacking a conjunction, and reacting to an attack against a conditional, gives rise to a new branch: a new play. By contrast, *Proponent's* choices do not generate new plays; and neither do *Opponent's* choices for quantifiers (defending an existential quantifier, attacking a universal quantifier).

The participants **P** and **O** of the dialogues that we are here interested in - the dialogues used for characterizing validity - are of course idealized agents. If real-life agents took their place, it might happen that one of the players was cognitively restricted to the point of following a strategy which would make him lose against some, or even every sequence of moves by the opponent - even if a winning strategy would be available to him. The idealized agents of the dialogues are not hence restricted: their 'having a strategy' means simply that there exists, by combinatorial criteria, a certain kind of function; it does not mean that the agent possesses a strategy in any cognitive sense.

Plays of a dialogue are sequences of dialogically signed expressions, and they share their first member, the *thesis* of the dialogue. In particular, plays can always be analyzed into dialogical games: any play is of the form $\Delta_1 \dots \Delta_n$, where the Δ_i are dialogical games ($i := 1, \dots, n$). The members of plays other than the thesis are termed *moves*. A move is either an attack or a defence.

The particle rules stipulate exactly which moves are to be counted as attacks. Exactly those moves $X\text{-}Y\text{-}e$ whose expression component e is a first-order formula, are said to have *propositional content*. Recall that in the case of conditional and negation some moves with propositional content count as attacks. (In the actual design of a dialogue there usually is a notational device to differentiate between those moves with propositional content that are attacks and those that are not.)

We move on to introduce a number of structural rules for dialogues designed for the language $L[\tau]$. We will write $D(A)$ for the dialogue about A , i.e. the dialogue whose thesis is A . Further, we will write $\Delta[n]$ for the member of the sequence Δ with the position n . Let A be a first-order sentence of vocabulary τ . We have the following structural rules (SR-0) to (SR-6) regulating plays τ in $\Delta \in D(A)$, i.e. members of the dialogue $D(A)$

(SR-0) (Starting rule)

- a) The dialogically signed expression $\langle \mathbf{P}\text{-!}\text{-}A \rangle$ belongs to the dialogue $D(A)$: the thesis A stated by *Proponent* is itself a play in the dialogue about A .
- b) If Δ is any play in the dialogue $D(A)$, then the thesis A has position 0 in Δ . If $\Delta \in D(A)$, then

$$\Delta[0] = \langle \mathbf{P}\text{-!}\text{-}A \rangle.$$

- c) At even positions **P** makes a move, and at odd positions it is **O** who moves. That is, each $\Delta[2n]$ is of the form $\langle \mathbf{P}\text{-}Y\text{-}B \rangle$ for some $Y \in \{?, !\}$ and $B \in \text{Sub}(A)$; and each $\Delta[2n+1]$ is similarly of the form $\langle \mathbf{O}\text{-}Y\text{-}B \rangle$. Every move after $\Delta[0]$ is a reaction to an earlier move made by the other player, and is subject to the particle rules and the other structural rules.

(SR-1.I) (Intuitionistic round closing rule).

Whenever player X has a turn to move, he may attack any (complex) formula asserted by his opponent, Y , or he may defend himself against the *last not already defended attack* (i.e. the attack by Y with the greatest associated natural number such that X has not yet responded to that attack).

A player may postpone defending himself as long as he can perform attacks. Only the latest attack that has not yet received a response may be answered: If it is X 's turn to move at position n ,

and positions l and m both involve an unanswered attack ($l < m < n$), then player X may *not* at position n defend himself against the attack of position l .

(SR-1.C) (Classical round closing rule)

Whenever player X has turn to move, he may attack any (complex) formula asserted by his opponent, Y , or he may defend himself against any attack, including those which have already been defended. That is, here even redoing earlier defences is allowed.

(SR-2) (Branching rule for plays)

If in a play $\Delta \in \mathcal{D}(A)$ it is \mathbf{O} 's turn to make a propositional choice, that is, to defend a disjunction, attack a conjunction, or react to an attack against a conditional, then Δ extends into two plays $\Delta_1, \Delta_2 \in \mathcal{D}(A)$,⁴

$$\Delta_1 = \Delta \overset{\phi}{\alpha} \text{ and } \Delta_2 = \Delta \overset{\phi}{\beta}$$

differing in the chosen disjunct, conjunct *resp.* reaction, α vs. β . More precisely: Let $\{n \leq \max\{m : \Delta[m]\}\}$.

- If $\Delta[n] = \langle \mathbf{O} \text{--} ! \text{--} B \vee C \rangle$ and $\Delta[\max] = \langle \mathbf{P} \text{--} ? \text{--} \vee \rangle$, then

$$\alpha := \langle \mathbf{O} \text{--} ! \text{--} B \rangle \text{ and } \beta := \langle \mathbf{O} \text{--} ! \text{--} C \rangle.$$

- If $\Delta[n] = \Delta[\max] = \langle \mathbf{P} \text{--} ! \text{--} B \wedge C \rangle$, then

$$\alpha := \langle \mathbf{O} \text{--} ! \text{--} L \rangle \text{ and } \beta := \langle \mathbf{O} \text{--} ! \text{--} R \rangle.$$

- If $\Delta[n] = \langle \mathbf{O} \text{--} ! \text{--} B \rightarrow C \rangle$ and $\Delta[\max] = \langle \mathbf{P} \text{--} ? \text{--} B \rangle$, then

$$\alpha := \langle \mathbf{O} \text{--} ? \text{--} * \rangle \text{ and } \beta := \langle \mathbf{O} \text{--} ! \text{--} C \rangle.$$

where $*$ is an attack marker corresponding to the logical form of the formula B .

No moves other than propositional moves made by \mathbf{O} will trigger branching.

(SR-3) (Shifting rule)

When playing a dialogue $\mathcal{D}(A)$, \mathbf{O} is allowed to switch between ‘alternative’ plays $\Delta, \Delta' \in \mathcal{D}(A)$. More exactly, if \mathbf{O} loses a play Δ , and Δ involves a propositional choice made by \mathbf{O} , then \mathbf{O} is allowed to continue by switching to another play - existing by the *Branching rule* (SR-2). Concretely this means that the sequence $\Delta \cap \Delta'$ will, then, be a play, i.e. an element of $\mathcal{D}(A)$.

It is precisely the *Shifting rule* that introduces plays which are not plain dialogical games. (Dialogical games are a special case of plays: they are identified with unit sequences of dialogical games.)

As an example of applying the *Shifting rule*, consider a dialogue $\mathcal{D}(A)$ proceeding from the hypotheses (or *initial concessions* of \mathbf{O}) $B, \neg C$, with the thesis $A := B \wedge C$. If \mathbf{O} decides to attack the left conjunct, the result will be the play

$$(\langle \mathbf{P} \text{--} ! \text{--} B \wedge C \rangle, \langle \mathbf{O} \text{--} ? \text{--} L \rangle, \langle \mathbf{P} \text{--} ! \text{--} B \rangle)$$

⁴) If $\bar{a} = (a_0, \dots, a_n)$ is a finite sequence and a_{n+1} is an object, $\bar{a} \cap a_{n+1}$ is by definition the sequence $(a_0, \dots, a_n, a_{n+1})$. If $\bar{a} = \bar{a}_1 \cap \bar{a}_2$, then, \bar{a}_1 is said to be an initial segment of \bar{a} , and, if the sequence \bar{a}_2 is not empty, then we say that \bar{a}_1 is the initial proper segment of \bar{a} .

and **O** will lose (because he has already conceded B). But then, by the *Shifting rule*, **O** may decide to do have another try. This time he wishes to choose the right conjunct. The result is the play

$$(<\mathbf{P}\text{!-}B\wedge C>, <\mathbf{O}\text{?-}L>, <\mathbf{P}\text{!-}B>, <\mathbf{P}\text{!-}B\wedge C>, <\mathbf{O}\text{?-}R>, <\mathbf{P}\text{!-}C>)$$

Observe that this play consists of two dialogical games, namely

$$(<\mathbf{P}\text{!-}B\wedge C>, <\mathbf{O}\text{?-}L>, <\mathbf{P}\text{!-}B>) \text{ and } <\mathbf{P}\text{!-}B\wedge C>, <\mathbf{O}\text{?-}R>, <\mathbf{P}\text{!-}C>$$

By contrast, this play is not itself a dialogical game.

(SR-4) (Winning rule for plays)

A play $\Delta \in \mathcal{D}(A)$ is *closed*, if $\Delta = (\Delta_1, \dots, \Delta_n)$, where the Δ_i are dialogical games, and in the most recent dialogical game Δ_n there appears the same positive literal in two positions, one stated by X and the other one by Y. That is, Δ is closed if for some $k, m < \omega$ and some positive literal $\ell \in \text{Sub}(A) \cup \{A\}$, we have:

$$\Delta_n[k] = \ell = \Delta_n[m]$$

where $k < m$ and furthermore, k is odd if, and only if m is even or equal to zero. If this condition is not satisfied, Δ is *open*.

If a play is closed, the player who stated the thesis (that is, **P**) *wins* the play; otherwise he loses it. A play is *finished*, if it is either closed, or else such that no further move is allowed by the particle rules or (other) structural rules. If a play is finished and open, **O** wins the play. Observe that whenever a play $\Delta \in \mathcal{D}(A)$ is finished, there is no further play $\Delta' \in \mathcal{D}(A)$ such that Δ is an initial segment of Δ' .

(SR-5) (Formal use of atomic formulae)

P cannot introduce positive literals: any positive literal must be stated by **O** first. Positive literals cannot be attacked.

In the following, when introducing material dialogues we will consider too, when speaking of First-order logic, intuitionistic dialogues with additional hypotheses introduced as *initial concessions* by **O**, such as:

$$\forall x_1 \dots \forall x_n (Rx_1 \dots x_n \vee \neg Rx_1 \dots x_n)$$

where R is a relation symbol of a fixed vocabulary τ . That is, the relevant hypotheses are instances of (a universal closure of) *tertium non datur*. In the presence of such hypotheses, we may use a more general formulation of the rule **(SR-5)**:

(SR-5*)

P cannot introduce literals: any literal (positive or not) must be stated by **O** first. Positive literals cannot be attacked.

Before we can state the structural rule (SR-6), or the *No delaying tactics rule*, we need some definitions.

DEFINITION [Strict repetition of an attack / a defence]

- a) We speak of a **strict repetition of an attack**, if a move is being attacked although the same move has already been challenged with the same attack before. (Notice that even though choosing the same constant is a strict repetition, the choice of ?-L and ?-R are in this context different attacks.)

In the case of moves where a universal quantifier has been attacked with a new constant, the following type of move must be added to the list of strict repetitions:

1. **A universal quantifier move is being attacked using a new constant, although the same move has already been attacked before with a constant which was new at the time of that attack.**
 2. **A universal quantifier move is being attacked using a constant that is not new, although the same move has already been attacked before with the same constant.**
- b) We speak of a **strict repetition of a defence**, if a challenging move (attack) m_1 , which has already been defended with the defensive move (defence) m_2 before, is being defended against the challenge m_1 once more with the same defensive move. (Notice that the left part and the right part of a disjunction are in this context two different defences.)
- c) In the case of moves where an existential quantifier has been defended with a new constant, the following type of move must be added to the list of strict repetitions:
1. **An attack on an existential quantifier is being defended using a new constant, although the same quantifier has already been defended before with a constant which was new at the time.**
 2. **An attack on an existential quantifier is being defended using a constant that is not new, although the same quantifier has already been defended before with the same constant**
- Notice that according to these definitions, neither a new defence of an existential quantifier, nor a new attack on a universal quantifier, represents a strict repetition, if it uses a constant that is not new but is however different from the one used in the first defence (or in the first attack) that was new.

(SR-6) ('No delaying tactics' rule) This rule has two variants, classical and intuitionistic, depending on whether the dialogue is played with the classical structural rule (SR-1.C), or with the intuitionistic structural rule (SR-1.I).

Classical: No **strict** repetitions are allowed.

Intuitionistic: If **O** has introduced a new atomic formula which can now be used by **P**, then **P** may perform a repetition of an attack. No other **strict** repetitions are allowed.

DEFINITION [Validity] *A first-order sentence A is said to be **dialogically valid** in the classical (intuitionistic) sense, if all plays belonging to the classical (resp. intuitionistic) dialogue $D(A)$ are closed.*

It is possible to prove that the dialogical definition of validity coincides with the standard definition, both in the classical and in the intuitionistic case. First formulations of the proof were developed in the PhD-Thesis by Kuno Lorenz (reprinted in Lorenzen/Lorenz 1978), Haas (1980) and Felscher (1985) proved the equivalence for intuitionistic First-order logic (by proving the correspondence between intuitionistic dialogues and intuitionistic sequent calculi); while Stegmüller (1964) established the equivalence in the case of classical First-order logic. Rahman (1994: 88-107), who stressed the idea that dialogues for validity could be seen as a proof-theoretical frame to build tableaux systems, proved directly the equivalence between the two types of dialogues and the corresponding semantic tableaux, from which the result extends to the corresponding sequent calculi.

Philosophical remarks: propositions as games.

Particle rules determine dynamically how to extend a set of expressions from an initial assertion. In the game perspective, one of the more important features of these rules is that they determine, whenever there is a choice to be made, who will choose. This is what can be called the pragmatic dimension of the dialogical semantics for the logical constants. Indeed, the particle rules can be seen as a proto-semantics, i.e. a game scheme for a not yet determined game which when completed with the appropriate structural rules will render the game semantics, which in turn will build the notion of validity.

Actually by means of the particle rules games have been assigned to sentences (that is, to formulæ). But sentences are not games, so what is the nature of that assignment? The games associated to sentences are meant to be *propositions* (i.e. the constructions grasped by the (logical) language speakers). What is connected by logical connectives are not sentences but propositions. Moreover, in the dialogic, logical operators do not form sentences from simpler sentences, but games from simpler games. To explain a complex game, given the explanation of the simpler games (out) of which it is formed, is to add a rule which tells how to form new games from games already known: if we have the games A and B , the conjunction rule shows how we can form the game $A \wedge B$ in order to assert this conjunction.

Now, particle rules have another important function: they not only set the basis of the semantics, and signalise how it could be related to the world of games – which is an outdoor world if the games are assigned to prime formulæ, but they also show how to perform the relation between sentences and propositions. Sentences are related to propositions by means of assertions, the content of which are propositions. Assertions are propositions endowed with a theory of force, which places logic in the realm of linguistic actions. The forces performing this connection between sentences and propositions are precisely the attack (?) and the defence (!). An attack is a demand for an assertion to be uttered. A defence is a response (to an attack) by acting so that you may utter the assertion (e.g. that A). Actually the assertion force is also assumed: utter the assertion that A only if you know how to win the game A .

Certainly the "know" introduces an epistemic moment, typical of assertions made by means of judgements. But it does not presuppose in principle the quality of knowledge required. The constructivist moment is only required if the epistemic notion is connected to a tight conception of what means that the player X *knows that there exists* a winning game or strategy for A .

Let us take examples of dialogues, classical and intuitionistic.

EXAMPLE: Consider the classical dialogue $D(p \vee \neg p)$. Its thesis is $p \vee \neg p$, where p is an atomic sentence. In Figure 1, a dialogical game from dialogue $D(p \vee \neg p)$ is described. This dialogical game is won by **P**:

O			P		
				$p \vee \neg p$	
1	$?-\vee$			$\neg p$	
3	p			—	
[1]	$[?-\vee]$	0]		p	

II.1.3.f1. Classical rules, **P** wins.

The outer columns indicate the position of the move inside the dialogical game, while the inner columns state the position of the earlier move which is being attacked. The defence is written on the same line with the corresponding attack: an attack together with the corresponding defence constitutes a so-called *closed round*. The sign ‘—’ indicates that there is no possible defence against an attack on a negation.

In the dialogical game of the example, **P** wins because after **O**'s last attack in move 3, **P** is allowed - according to the classical rule SR-1.C - to defend (once more) himself against **O**'s attack made in move 1, which was certainly *not the last attack* of **O**, and so the game in question is closed. **P** states his new defence in move 4. (Actually **O** does not repeat his attack of move 1: what we have written between square brackets simply serves to remind of the attack against which **P** is re-acting.)

- In fact the described dialogical game is the only finished play of the dialogue $D(p \vee \neg p)$: **O** could not prolong the play any further by making different moves. Hence not only does **P** win the described particular dialogical game - in fact he has a winning strategy in the dialogue, i.e. he is able to win no matter what **O** does. In other words, the sentence $p \vee \neg p$ is dialogically valid in the classical sense (cf. Definition *Def:validity*).

Here is an example concerning Peirce's Law and which requires to consider two plays:

EXAMPLE:

In the version of strategy dialogues what actually happens is that **O** generates two dialogical games one defending and the other counterattacking. Both dwill be closed and thus won by **P**.

O			P		
				$((p \rightarrow q) \rightarrow p) \rightarrow p$	0
1	$(p \rightarrow q) \rightarrow p$	0		p	4
			1	$p \rightarrow q$	2
2	p	2			

II.1.3.f2. Classical rules, **P** wins.

O			P		
				$((p \rightarrow q) \rightarrow p) \rightarrow p$	0
1	$(p \rightarrow q) \rightarrow p$	0		p	4
3	p		1	$p \rightarrow q$	2

II.1.3.f3 Classical rules, **P** wins.

Actually this produces a play with two dialogical games. Let us label each dialogical game with a roman letter and put all in only one graphic. In the graphic below we splitted the play in two showing the dialogical games produced - simpler would be to eliminate the outer columns and add the label directly to the formulae, but this notation will make it easier to show the relation to (the branches produced by a correspondent) sequent calculi.

The expression between the signs ‘<’ and ‘>’ signalise that the Opponent has decided in the choice I.2 not to counterattack the expression inside those signs but defend himself. This expression is then not at stake in the play I.2 and can be considered as an *attack marker* rather than a formula

Because of the tree-like structure of the proof we will assume that the thesis (move 0), the first challenge of **O** (move 1), which occur before the splitting takes place, and the answer (move 4) are shared by both dialogical games and will neither repeat them:

O			P		
				$((p \rightarrow q) \rightarrow p) \rightarrow p$	0
1	$(p \rightarrow q) \rightarrow p$	0		p	I.4 II.4
I.3	p		1	$\langle p \rightarrow q \rangle$ $p \rightarrow q$	I.2 II.2
	II.3	p	II.2		

II.1.3.f4 Classical rules, **P** wins.

Let us consider now the intuitionistic variant of the dialogue of the first example.

EXAMPLE: In figure below, a dialogical game from the intuitionistic dialogue $D(p \vee \neg p)$ is described. This game is won by **O**:

O			P		
				$p \vee \neg p$	
1	$\neg \vee$	0		$\neg p$	
3	p	2		—	

II.1.3.f5. Intuitionistic rules, **O** wins.

It is **O** who wins the dialogical game of the example: the game is open, and no further move is possible following the intuitionistic structural rules. In particular remaking an earlier move (i.e., answering to an attack which was *not the last one* - as in the above example of a classical dialogue - is not possible.

In fact **O** has trivially a winning strategy in the intuitionistic dialogue $D(p \vee \neg p)$: **P** cannot prevent, by making different moves, **O** from generating precisely the described play won by **O**.

- Observe, in particular, that the sentence $p \vee \neg p$ is not dialogically valid in the intuitionistic sense. (This does not mean, of course, that thereby the sentence $\neg(p \vee \neg p)$ would be intuitionistically valid!)

The following example shows the fail of double negation in intuitionistic logic

EXAMPLE

$D(\neg \neg p \rightarrow p)$

O			P		
				$\neg \neg p \rightarrow p$	
1	$\neg \neg p$				
	—			$\neg p$	
3	p	2		—	

II.1.3.f6. Intuitionistic rules, **O** wins.

O wins because **P** is not allowed to use the atomic formula stated by **O** at move 3 to defend the challenge of move 1. Indeed, move 3 is the last attack of **O** and **P** must answer now to this attack. Unfortunately, by the particle rule of negation, there is no defence to challenged negation. Only counterattacks are possible. But p is an atomic formula which cannot be counterattacked!

To come back to a success story for **P** let us see a more trickier case. Namely, an intuitionistic dialogue for $D(\neg \neg(p \vee \neg p))$ where **P** should have a winning strategy. Indeed, the double negation of any valid classical formula is valid intuitionistically too!

EXAMPLE

$D(\neg \neg(p \vee \neg p))$

O			P		
				$\neg\neg(p\vee\neg p)$	
1	$\neg(p\vee\neg p)$			—	
	—			$p\vee\neg p$	
3	?- \vee			$\neg p$	
5	p			—	
	—			$p\vee\neg p$	
7	?- \vee			p	

II.1.3.f7. Intuitionistic rules, **P** wins.

The tricky point is move 6 where **P** is allowed to repeat the attack on the first move of **O** because since move 1, **O** introduced a new atomic formula (see SR-6)S. Indeed at move 5 **O** introduced the positive literal p and this can be now used to defend the new occurrence of the disjunction.

The way to build a winning strategy for dialogues for first-order logic is not really different from the propositional case: Here the Proponent will try to wait so long as he can before choosing a value for the variables. More precisely, he will wait until the Opponent has chosen first the value for the variables at stake and later on he will simply *copy-cat* them. Let us show examples of dialogues for first order logic:

EXAMPLE

$D((\forall x((Ax\vee Bx)\wedge\neg Ax))\rightarrow\forall x(\neg\neg Bx\vee Cx))$

O			P		
				$(\forall x((Ax\vee Bx)\wedge\neg Ax))\rightarrow\forall x(\neg\neg Bx\vee Cx)$	
1	$\forall x((Ax\vee Bx)\wedge\neg Ax)$?- $\forall x(\neg\neg Bx\vee Cx)$	
3	?- $\forall x/k$			$\neg\neg B_k\vee C_k$	
5	?- \vee				
7	$(A_k\vee B_k)\wedge\neg A_k$?- $\forall x/k$	
9	$A_k\vee B$?-L	
11	A_k			?- \vee	0
13	$\neg A_k$?-R	2
	—		3	A_k	4

II.1.3.f7. Classical rules, **P** wins.

Validity on frames: the dialogical approach to modal logic

Introduction to modal dialogic

Modal dialogic is a systematic account of an explicit notion of context, in the sense that the latter is introduced by an explicit label. Modal moves are hence dialogical expressions with a supplementary label, indicating the context in which the move has been made. The usual modal operators are then defined in the following way:

\Box, \Diamond	<i>Attack</i>	<i>Defence</i>
$X \Box A \ i$ ($\Box A$ has been stated by player X at context i)	$Y ? \Box_j (iR^D_j) i$ (at the context i the challenger Y attacks by choosing a dialogically accessible context j)	$X A \ j$ (the defender claims that A holds at the label j)
$X \Diamond A \ i$ ($\Diamond A$ has been stated by player X at context i)	$Y ? \Diamond i$ (at the context i the challenger Y attacks asking X to choose a j where A holds)	$X A \ j (iR^D_j)$ (the defender chooses the context j such that j is dialogically accessible from i)

V.3.t1

[add states of the game for modal dialogic]

In modal dialogic the frame conditions implemented as special structural rules which allow the Proponent to increase his choice possibilities while challenging a necessity operator or defending a possibility operator.

DEFINITION

If at i the Opponent while challenging a necessity operator or defending a possibility operator chooses a new label j such that i is a proper initial segment of j we say that the Opponent has *introduced* j and *conceded* that the label j is *dialogically accessible* from the label i (for short $iR^D j$):

\Box, \Diamond	<i>Attack</i>	<i>Defence</i>
$P \Box A \ i$ ($\Box A$ has been stated by P at context i)	$O ? \Box_j (iR^D_j) i$ (at the context i O <i>introduces</i> j and <i>concedes</i> $iR^D j$)	$P A \ j$
$O \Diamond A \ i$ ($\Diamond A$ has been stated by O at context i)	$P ? \Diamond i$ (at the context i P asks O to choose a j where A holds)	$O A \ j (iR^D_j)$ (at the context i O <i>introduces</i> j and <i>concedes</i> $iR^D j$)

V.3.t2

MODAL FORMAL RULE

At label **i** the Proponent may choose a label **j** such that $iR^D j$ iff **j** has been introduced by the Opponent before or this choice has been allowed by the appropriate modal structural rule.

\Box, \Diamond	<i>Attack</i>	<i>Defence</i>
$O \Box A \text{ i}$ ($\Box A$ has been stated by O at context i)	$P ? \Box_j (iR^D_j) i$ at the context i P chooses a j $iR^D j$ such that j has been introduced by O or has been allowed by the appropriate structural rule)	$O A \text{ j}$
$P \Diamond A \text{ i}$ ($\Diamond A$ has been stated by P at context i)	$O ? \Diamond i$ (at the context i O asks P to choose a j where A holds)	$PA \text{ j } (iR^D_j)$ (at the context i P chooses a j $iR^D j$ such that j has been introduced by O or has been allowed by the appropriate structural rule)

V.3.t3

LOGIC

MODAL STRUCTURAL RULES

K

No conditions

D

The Proponent may choose a label i though it has NOT been chosen by the Opponent before.

T

Assume that **P** is at i . **P** may then choose i .

B

Assume that **P** is at $i.n$. **P** may then choose $i.n$ and

P may then choose also i ($i.n$ is the immediate extension of i).

K4

Assume that **P** is at i . **P** may then choose a j such that i is an initial segment of j .

S4

Assume that **P** is at i . **P** may then choose i and he may choose j such that i is an initial segment of j .

For short an formulating both conditions at once: **P** may choose a j such that i is an initial segment (proper or otherwise) of j .

S4.3

For the reflexive and transitive cases take the structural rules for **S4**-frames. For the linear condition: Assume that **P** is at i and also

assume that **O** conceded that $iR^D j$ and $iR^D k$. **P** may then ask **O** to choose between conceding either $jR^D k$ or $kR^D j$.

K5

Assume that **P** is at $i.j$ **P** may then choose at that context $i.k$ and $i.j$.

S5.

Take the structural rules for the reflexive-, symmetric- and transitive-frames.

For short: Assume that **P** is at i . Then **P** can choose any (already introduced) label j (even $j=i$).

Dialogues for first order modal logic: varying and constant domains

Dialogues for Constant Domains:

Dialogues for constant domains are quite simple to develop: One just uses classical first-order logic without restrictions on the Proponent -choices of constants.

Actually, for reasons that will be clear below, we will assume that dialogues for first-order logic are defined with the help of two disjoint sets of free variables called the set **K** of k-terms, and the set **P** called parameters, that will never be bounded by quantifiers. The particle rules are defined in the union of both sets, but strategically, while proving validity, we will assume that the opponent will always chose (when he can), new elements of **P** and the proponent will make his choices in the union of both sets.

Dialogues for Varying Domains:

For dialogues corresponding to globally varying domain semantics within the free logic approach there is one change needed: we introduce a whole family of lists of parameters, one for each label, rather than a single list of parameters. More specifically, we assume that to each context-label *i* there is associated an infinite list of parameters, in such a way that different context-labels never have the same parameter associated with them.

Thus, we write $pi.i$ to indicate that p is a parameter associated with $wi.i$

Actualist Structural Rule for globally varying domains: the free logic approach

- Choices for quantifiers stated at wi by any player have to be chosen from the set of Pi of parameters associated to wi .

Note that the quantifiers are understood as actualist: That is, the range of their variables does neither go beyond the context w where these quantifiers have been stated nor does it extend to the set **K**. Thus, neither

$\forall xAx \rightarrow Ak$, nor

$Ak \rightarrow \exists xAx$

will be valid. Indeed any dialogue starting with them will force the Proponent to choose a parameter and thus he will not be able to produce the required atomic formula Ak .

Exercise:

Prove as an exercise that neither the Barcan- nor the converse Barcan-formulae are valid

Actualist Structural Rule for monotonic varying domains: the free logic approach

- Choices for quantifiers stated at $wi.i...$ have to be chosen from the set of $Pi.i...$ or $Pi...$ where i is an initial fragment of i (that is, $wi.. R wi.i...$).

Exercise:

Prove that

the converse Barcan-formulae are valid

the Barcan-formulae are non valid
the free logic formulae are non valid

Take the following formulation

The Proponent might choose π_i at w_i if, π_i has been **already asserted** (by means of a challenge to a universal quantifier or of a defence to an existential quantifier) at w_i or if it is completely new in the dialogue.

Does this correspond to one of the logics described above? Why?

Actualist Structural Rule for anti-monotonic varying domains: the free logic approach

- Choices for quantifiers stated at $w_i...$ have to be chosen from the set of $Pi.i...$ or $Pi...$ where i is an initial fragment of $i.i$ (that is, $w_i.. R w_i.i...$).
(Note that here that this is about the choices at $w_i..$ not at $w_i.i$ as in the monotonic case)

Exercises

Prove that

- the converse Barcan-formulae are non-valid
- the Barcan-formulae are valid
- the free logic formulae are non valid
- give a formulation for monotonic and anti-monotonic varying domains without free logic.

Soundness of first-order positive free logic \mathbf{K}

When we say in metalogic that we prove that a given proof system is “sound” we mean that with this system we cannot prove any formula it should not. For example; if our dialogical proof system for \mathbf{K} were not sound then we would be able to prove some formula, such as $\Box A \rightarrow A$, which is beyond of the frame validity characterizing \mathbf{K} . More precisely, to say that our dialogical proof system is sound means: if a formula ϕ has a dialogical proof for \mathbf{K} ; then this formula is valid in the logic \mathbf{K} as described by the model theoretical characterisation of \mathbf{K} -validity.

To prove this we need some previous work:

- **Dialogical tree:** For the sake of simplicity we will assume a tree-like rewriting of the dialogues. That is, instead of the notation

O			P		
				iX	0
1	iV			$i.nZ$	4
3	$i.nY$			$i.W$	2

We will write

- 0 iPX
- 1 iOV
- 2 iPW
- 3 $i.nOY$
- 4 $i.nPZ$

- If the application of the appropriate rule forces **P** to state an atomic formula in the dialogue and he can not, then we will anyway write them down but between square brackets.
E.g. if the formula to be extended is $i(\mathbf{P}pvq)$, and **O** did not assert before neither ip nor iq , then we will extend the branch in the following way

$i[\mathbf{P}p]$
 $i[\mathbf{P}q]$

The square brackets should indicate that though **P** can not play them in the dialogue, they will obtain a valuation in the model (see below) , namely

$$v_{f(i)} \neg p = 1 \text{ and } v_{f(i)} \neg q = 1$$

If the Proponent answers with a box such as $[\mathbf{P}a]$, he can play a further move. Indeed, as mentioned before, the box means the Proponent can not really play that move in the standard dialogue.

$n \quad i\mathbf{O} \Box b$
 $n+1 \quad i\mathbf{P} \Box a$
 $n+2o \quad i.\mathbf{O}? (n+1) i.l \Box \quad (\text{the Opponent challenges line } n+1 \text{ choosing } i.l)$
 $n+3 \quad [i.l\mathbf{P}a]$
 since the move $n+3$ is a box the Proponent may move
 $n+4 \quad i.\mathbf{P}? (n) i.l.l \Box$
 $n+5 \quad i.l\mathbf{O}b$

If the branch is closed with an atomic formula, say a , that has been played by the Opponent after the move $[\mathbf{P}a]$, then we will assume a further move where the Proponent opens the box, though this moves will not have a new move-number: it is the move closing the branch.

Furthermore in the cases of the connectives where the Proponent has a choice (defending a disjunction and when challenging a conjunction), both choices will be played without delay one after the other

Notice that dialogical trees are not Beth(Smullyan)-Tableaux though the resulting proof might be indistinguishable from those Tableaux . Indeed the semantics of a logical constant in a Tableau-System is defined by the **T (O)**-rules and the **F (P)**-rules, but the semantics of a logical constant in a dialogue is primarily defined by the so-called particle rules. Particle-rules should be player independent: the difference between **T (O)**-rules and the **F (P)**-rules is a result of the strategical level. This makes it that we could formulate tonk-particles for Tableaux but not for dialogues. See exercise 2 at the end of the soundness proof.

Definition 7 [Satisfiable in varying domain models]:

Let us consider a set S of signed (and labelled) formulae, where members of S may contain parameters and k-terms. We say that S is satisfiable in the varying domain model $\mathbf{M} = \langle W, R, D, i \rangle$ with respect to an assignment \mathbf{g} if there is a mapping \mathbf{f} assigning to each label i of a formula in the branch some possible world $\mathbf{f}(i)$ in W such that:

1. If i and $i.n$ both occur as labels in S , then $\mathbf{f}(i)R\mathbf{f}(i.n)$ in \mathbf{M} .

2. If iXA is in S , (where X signalises that the formula is **O**- or **P**-signed), then $v_{f(i)} * g(A) = 1$ in M . In words A is true at the world $f(i)$ of the model M with respect to the assignment g – where:
 $v_{f(i)} * g(OA) = v_{f(i)} * g(A) = 1$,
 $v_{f(i)} * g(PA) = v_{f(i)} * g(\neg A) = 1$ and
 $v_{f(i)} * g([PA]) = v_{f(i)} * g(\neg A) = 1$ (where A is atomic – see the comment in the chapter on the propositional case)
3. If the parameter pi occurs in S , then $g_{f(i)}(pi) \in D(f(i))$
4. If a free variable x (including a k-term) **other** than a parameter occurs in S , then $g_{f(i)}(x) \in D_F$

The rest carries over directly from the propositional case:

Since S has been restricted to a set of formulae the following kind of moves (of any player) will not be mapped into the model:

- ? $\vee i$,
- ? $\neg \wedge i$,
- ? $\neg \forall x/k$,
- ? \exists

- We say that a branch of a dialogue (produced by the shifting rule) is satisfiable if the set of labelled signed formulae on it is satisfiable in some model with respect to some valuation.
- We say that a dialogue (dialogical game) is satisfiable if some branch of it is satisfiable

Soundness lemma Q1 (SL-Q1):

A closed dialogue for varying domains (a dialogue won by **P**) is not satisfiable

PROOF:

- Suppose that we had a dialogue that was both closed and satisfiable.
- Since it is satisfiable, some branch of it is. Let S be the set of formulae on that branch and let it be satisfiable in the model M by means of the mapping f .
- Since the dialogue is closed (won by **P**) then for some labelled atomic formula A we must have iOA and iPA . But then both $v_{f(i)}(A) = 1$ and $v_{f(i)}(\neg A) = 1$ must be the case in M but this is not possible.

Soundness lemma Q2 (SL-Q2):

If (a section of) a dialogue for varying domains is satisfiable and a branch (produced by the shifting rule) of that (section of) dialogue is extended by appropriate particle rules, the result is another satisfiable (section of) a dialogue.

PROOF:

Let D be a (section of a) satisfiable dialogue and let B be the branch that is extended.

The steps to be considered are exactly the same steps as before though we must add the quantifier cases.

T

Assume that S on B is satisfiable in the varying domain model $M = \langle W, R, D, i \rangle$ with respect to an assignment g and the mapping f

Let us take the case of $iO\exists x\phi(x)$. If we apply the correspondent rule we will produce the branch **B1** containing the formulae:

$$iO \phi(pi) ([iO \phi(pi)]),$$

where pi is a parameter that is new to the branch

We must show that **B1**- that consists of the formulae of S and $iO\phi(pi)$ is still satisfiable

Since **B** is by hypothesis satisfiable in M with the mapping f and assignment g , and $iO\exists x\phi(x)$ is on **B**, by definition **IX-7** we have

$$v_{f(i)} * g (O\exists x\phi(x)) = v_{f(i)} * g (\exists x\phi(x)) = 1$$

Thus for some x -variant g' of g we have

$$v_{f(i)} * g' (\phi(x)) = 1$$

Define a new assignment g'' in the following way: for each variable (including parameters),

g'' is the same as g in all variables except x .

$$g''(pi) = g'(x).$$

Notice that since g' is an x -variant at $f(i)$, then $g'(x)$ and $g''(pi)$ is in $D(f(i))$.

Now, g'' and g agree on all variables except pi and pi is new to the branch **B**, hence it does not occur in any labelled formula on **B**.

Since the set of labelled formulae on **B** was satisfiable in the model with respect to the assignment g , this is also the case if we use the assignment g'' (once more: g and g'' differ only on pi but pi is not on the branch!) (see lemma IX.1)

Since g is an x -variant, g and g' agree on all variables except for x ;

Recall that g'' and g agree on all variables except pi

Therefore, g'' and g' agree on variables except for x and pi

But pi does not occur neither in $\phi(x)$ nor in the branch (because it was new).

Thus, g'' and g' agree on all the free variables of $\phi(x)$ except for x and by definition of $g''(pi) = g'(x)$

Now we can use the lemma IX.2 that says that one variable can be substituted by another provided the valuations are adjusted adequately

$$M, v_{f(i)} * g'(\phi(x)) = 1 \Leftrightarrow M, v_{f(i)} * g''(\phi(pi)) = 1$$

Since we know that $M, v_{f(i)} * g'(x) = 1$ is the case, then we have $M, v_{f(i)} * g''(pi) = 1$

And so we finished with the existential case

I leave the universal case for the reader

Soundness theorem:

If \mathbf{P} has a winning strategy for A using the \mathbf{K} -structural rules, A is valid.

PROOF: Assume that \mathbf{P} has a winning strategy for A using the \mathbf{K} -structural rules, but A is not \mathbf{K} -valid. We show that from this a contradiction follows.

Since \mathbf{P} has a winning strategy for A using the \mathbf{K} -structural rules there is a closed dialogue D that starts with $0\mathbf{P}A$. Thus, the first section of D is D_0 that consists in the thesis $0\mathbf{P}A$. The following sections of D are constructed by extending D_0 .

Since A is not \mathbf{K} -valid, there is a world w_i in some model M at which A is not true. Let $f(1)=w_i$, using this model and mapping $\{0\mathbf{P}A\}$ is satisfiable. Thus D_0 is satisfiable, since the set of formulae on its only branch is satisfiable

Since D_0 is satisfiable by lemma SL-Q2 so is any dialogue D we get that starts with D_0 and results by extending D_0 .

It follows that D is satisfiable.

D is closed by hypothesis, and this is impossible by SL-Q1.

Quod erat demonstrandum, 2006.